Supplementary Material

Proof of lemma 1

Let $B \in \mathbb{R}^{d\times r}$ a matrix such that $B^T B = I_r$ and span(B) is a dimension reduction subspace for the regression of Y on X . Let $B_a = [B, B_\perp] \in \mathbb{R}^{d\times d}$ where the columns of $B_\perp$ form an orthonormal basis of span$(B)^\perp$. Now assume $X \sim \mathcal{N}(0, I_d)$.

X has density $q(x) = \varphi_d(x) = (2\pi)^{-\frac{d}{2}} \exp(-\frac{||x||^2}{2})$ where φ_d is the d dimensional multinormal pdf. If x_B and $x_{B_\perp}$ are the orthogonal projections of x on span(B) and span$(B_\perp)$ respectively, then $x = x_B + x_{B_\perp} = B(B^T x) + B_\perp(B_\perp^T x)$ and $||x||^2 = ||x_B||^2 + ||x_{B_\perp}||^2 = ||B^T x||^2 + ||B_\perp^T x||^2$. Hence,

$$\begin{aligned} q(x) &= (2\pi)^{-\frac{d}{2}} \exp(-\frac{||x_B||^2 + ||x_{B_\perp}||^2}{2}) \\ &= (2\pi)^{-\frac{r}{2}} \exp(-\frac{||B^T x||^2}{2})(2\pi)^{-\frac{d-r}{2}} \exp(-\frac{||B_\perp^T x||^2}{2}) \\ &= \varphi_r(B^T x)\varphi_{d-r}(B_\perp^T x) \end{aligned}$$

and the quasi-optimal density is equal to

$$\tilde{q}_{r*}(x) = \frac{\pi_r(B^T x)\varphi_r(B^T x)\varphi_{d-r}(B_\perp^T x)}{P_{f,\epsilon}}$$

Now let $\tilde{X} \sim \tilde{q}_{r*}$ and consider the mapping $\tilde{X} \mapsto W = B_a^T \tilde{X}$. By a change of variable, for any continuous bounded function ψ from $\mathbb{R}^d$ to $\mathbb{R}^d$

$$\begin{aligned} \mathbb{E}(\psi(W)) &= \int_{\mathbb{R}^d} \psi(B_a^T \tilde{x})\tilde{q}_{r*}(\tilde{x})\mathrm{d}\tilde{x} \\ &= \underbrace{\frac{1}{|\det(B_a^T)|}}_{=1} \int_{\mathbb{R}^d} \psi(w)\tilde{q}_{r*}(B_a^{-T} w)\mathrm{d}w \end{aligned}$$

which implies that the density of W is

$$\begin{aligned} p_W(w) &= \tilde{q}_{r*}(B_a^{-T} w) && (1) \\ &= \frac{1}{P_{f,\epsilon}} \pi_r(B^T B_a^{-T} w)\varphi_r(B^T B_a^{-T} w)\varphi_{d-r}(B_\perp^T B_a^{-T} w) && (2) \\ &= \frac{1}{P_{f,\epsilon}} \pi_r(w_1)\varphi_r(w_1)\varphi_{d-r}(w_2) && (3) \end{aligned}$$

where $w^T = (w_1^T, w_2^T)$. By the same argument, if W has density $p_W(w) = \frac{1}{P_{f,\epsilon}}\pi_r(w_1)\varphi_r(w_1)\varphi_{d-r}(w_2)$ then $\tilde{X} = B_a^{-T} W \sim \tilde{q}_{r*}$. It is clear from (3) that to sample $W = (W_1^T, W_2^T)^T$ from p_W, it suffices to sample $W_2 \sim \mathcal{N}(0_{d-r\times 1}, I_{d-r})$, and $W_1 \sim p_{W_1}(w_1) = \frac{\pi_r(w_1)\varphi_r(w_1)}{P_{f,\epsilon}}$.