Extension of a new duality theorem in linear programming - Application to the breakdown of long-run marginal costs

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Summary

In some cases, the objective function of a linear programming problem can be split up into distinct elementary functions, all of which giving rise to a specific interpretation. The theorem we will reformulate was originally derived by Babusiaux (2000) in such a context. The aim was to define a marginal cost in CO_2 emission for each of the refined products of an oil refinery. The initial objective function resulted from the addition of a classic operating cost function and a CO_2-emission cost function. The author showed that at the optimum, under some assumptions, this marginal cost had an average-cost structure. In other words, the well-know duality property was extended to each one of the elementary functions, for which it is possible to define elementary dual variables. We reformulate and generalize this theorem. We show that it is valid in any basic feasible solution. Moreover, we provide a simple interpretation of this result. We then show, with a capital budgeting example, how to break down a long-run marginal cost into a marginal operating cost and a marginal equivalent investment cost. This decomposition allows us an in-depth analysis of the formation of long-run marginal costs. This theorem, with the associated concept of elementary dual variable, should give rise to a sizable number of applications in capital budgeting modeling.

1. Introduction

The objective function of a linear program can be made up of different elementary functions, each of these elementary functions capable of eliciting a specific interpretation. The theorem that we shall reformulate was initially derived by Babusiaux (2000) in connection with a problem of this type. The aim was to associate, with each finished petroleum product, a marginal cost linked to the CO_2 emissions of a refinery designed to process different crude oils. Roughly speaking, the aim was to minimize a total processing cost, while satisfying a demand constraint for each finished product. The total cost function to be minimized was composed of a first function representing the conventional costs associated with the purchase and processing of the different crude oils, and a second function corresponding to the cost linked to the CO_2 emission (proportional to the quantities of each crude oil processed). The author showed that at the optimum, with certain assumptions, the marginal costs associated with the CO_2 emission have an average-cost structure. Similarly, the marginal CO_2 contents of the different products have an average-content structure, in other words, they offer a distribution key per product of the refinery's emissions. This result is equivalent to generalizing the standard property of duality to the different elementary functions making up the objective function. We shall propose an extension of this theorem and demonstrate that it remains valid in any basic feasible solution (extreme point) of the problem analyzed (and not only at the optimum). We shall then offer an original interpretation, by showing that it amounts to breaking down the problem analyzed into a set of distinct sub-problems. Using a capital budgeting example, we shall then show how this theorem serves to break down a long run marginal cost into an a marginal operating cost and a marginal equivalent investment cost. Each marginal cost thus calculated has an average-cost structure. In capital budgeting, this breakdown helps to analyze the formation of long run marginal costs and provides useful information for a preliminary approach to sensitivity (without having to perform a complete sensitivity analysis).

2. Duality theorem in linear programming with a multi-component objective function

2.1 Statement of the theorem

Let us consider a linear program of which the objective function, which we shall call the total objective function, is the sum of several functions, which we shall call elementary objective functions. Let us consider any basic feasible solution of the program. It is possible to associate, with each elementary objective function, a vector of dual variables, which we shall call elementary dual variables. Each elementary dual variable vector satisfies a duality property: the product of this vector with the one formed of the right-hand side coefficients of the program is equal to the value of the corresponding elementary objective function. The total dual variable vector (associated with the total objective function) is equal to the sum of the elementary dual variable vectors.

This theorem represents a generalization, to the case of any feasible basic solution, of the theorem initially derived by Babusiaux in the specific case of an optimal solution.

2.2 Formalization of the theorem

Let us consider a linear program, written in its simplicial canonical form, of the type:

$Min\ F = CX$ where $X \in R^n$ (vector of which the components are the n variables of the model)
$AX = b$ A of format (m,n) and of rank m
$X \geq 0$

Let us break down the total function F into k elementary functions: $F = \sum_{i=1}^{k} F_i = \sum_{i=1}^{k} C_i X$

Each elementary function F_i can be associated with a vector Π_i of which the components are the m corresponding elementary dual variables. The duality formula is satisfied: $\Pi_i b = F_i$.

If Π denotes the total dual variable vector associated with F, we have: $\Pi = \sum_{i=1}^{k} \Pi_i$

2.3 Notations and demonstration

Let us consider a basic solution, not necessarily optimal, of the starting program. Let Y denote the vector formed of the m basic variables and $\overline{Y}$ the vector formed of the n-m null (nonbasic) variables.

$$X = \begin{bmatrix} Y \\ \overline{Y} \end{bmatrix}$$

Let B and $\overline{B}$ denote the matrices composing A, associated with all the subscripts of the basic and nonbasic variables respectively:

$$A = \begin{bmatrix} B\ \overline{B} \end{bmatrix}$$

By adopting the same approach for the vector C and the vectors C_i, with:

$$C = \begin{bmatrix} D\ \overline{D} \end{bmatrix},\ C_i = \begin{bmatrix} D_i\ \overline{D}_i \end{bmatrix}$$

With: $$D = \sum_{i=1}^{k} D_i$$

For the feasible solution considered, the value of the objective function is:

$$F = \begin{bmatrix} D\ \overline{D} \end{bmatrix} \begin{bmatrix} Y \\ \overline{Y} \end{bmatrix} = DY + \overline{D}\overline{Y} = DY$$

Moreover: $\begin{bmatrix} B\ \overline{B} \end{bmatrix} \begin{bmatrix} Y \\ \overline{Y} \end{bmatrix} = b \Rightarrow Y = B^{-1} b$

Hence: $F = DB^{-1}b$

The vector Π formed of the *m* total dual variables is given by: $\Pi = DB^{-1}$. The duality formula is written: $F = \Pi b$. By replacing *F* and *D*, the above equation is written:

$$\sum_{i=1}^{k} F_i = \left(\sum_{i=1}^{k} D_i \right) B^{-1}b$$

Or further: $$\sum_{i=1}^{k} \left(F_i - D_i B^{-1}b \right) = 0$$

By setting $\Pi_i = D_i B^{-1}$, we define for each elementary function a vector formed of the corresponding dual values (evidently with: $\sum_{i=1}^{k} \Pi_i = \Pi$). The duality formula is satisfied for each elementary function: $F_i = D_i Y = D_i B^{-1} b = \Pi_i b$.

2.4 Interpretation

It may be observed that Π_i represents the vector formed of the dual values (corresponding to the basic feasible solution concerned) for the following program:

$$\begin{array}{ll} Min & F_i = C_i X \\ AX = b & \\ X \geq 0 & \end{array}$$

In actual fact, for any demonstration, it would have sufficed to state that the feasible solution considered for the program initially described (with *F* for objective function) is also a feasible solution of the *k* distinct programs obtained by replacing *F* by F_i. For each of these programs, it is accordingly possible to determine a vector Π_i formed of the corresponding dual values and satisfying the duality formula[1]. The vector Π formed of the dual values corresponding to the initial program is then equal to the sum of the vectors Π_i (since by definition $\sum_{i=1}^{k} \Pi_i b = \sum_{i=1}^{k} F_i = F = \Pi b$).

From the modeling standpoint, this is equivalent to determining the optimal solution of the starting program, and then calculating the *k* vectors Π_i by considering this solution as a feasible solution of each of the *k* sub-programs having F_i as objective function. This procedure offers the advantage of enabling the interpretation of each dual value vector and of preserving the additivity of the dual values.

Note that we do not refer here to the results relative to the weak duality, i.e. to the existence of a non-null difference (except at the optimum) between the values of the objective functions of

[1] Note that the duality formula relates to any non-degenerate basic solution (not necessarily optimal) by virtue of the definition given of the dual variable vector in section 2.3. We shall return to this point further in this section.

a program and of its dual for any feasible solutions of each of the two programs. For any basic feasible solution, we thus define vectors Π_i satisfying the duality formula $\Pi_i b = C_i X$. For each of the *k* problems addressed, the vector Π_i does not generally constitute a feasible solution of the corresponding dual program. It will constitute a feasible solution if *X* is the optimal solution of the primal program considered.

3. Separation of a long-run marginal cost into a marginal operating cost and a marginal equivalent investment cost

A problem of linear programming can lead to the determination of the long run marginal costs including operating costs and investment costs. The objective function accordingly reflects these two types of cost. We propose to use the theorem presented in order to separate a long run marginal cost into a marginal operating cost and a marginal equivalent investment cost. Each of these two terms will have a mean cost structure. This breakdown provides a better interpretation of long run marginal costs and of their formation.

We can give an illustration thereof by taking the example, simplified to the extreme, of an unsophisticated oil refinery, qualified as *Topping Reforming*. Roughly speaking, such a refinery processes a given crude oil to produce three main types of finished product: gasoline, diesel and heavy fuel oil. The yields obtained for each of the products are assumed to be known. Over a typical year, the production of the refinery must satisfy the demand forecast for each of the finished products. Let us assume that to boost the gasoline yield of the refinery, it is possible to invest in a catalytic cracker called FCC. Such a unit serves to convert a portion of the heavy fuel oil into gasoline. Investing in this unit accordingly modifies the finished product yields of a portion of the crude oil processed by the refinery. The aim is to satisfy the demand constraints while minimizing the total annual costs (operating costs and equivalent investment cost).

To do this, we shall consider that the total quantity of crude oil processed by the refinery in a year results from the addition of two variables: the quantity (denoted *x*) processed according to the "topping/reforming" scheme and the quantity (denoted *y*) processed according to the "topping/reforming/cracking" scheme. It may in fact be optimal to divert to the cracking unit only heavy fuel oil derived from a limited portion of the total quantity of crude processed. Let us assume that the capacity of the cracking unit represents 25% of the quantity of crude oil following the "topping/reforming/cracking" scheme. Furthermore, the investment cost is assumed to depend linearly on the capacity of the FCC unit. By taking account of the company discount rate and the service life of the unit, we can consider that this investment corresponds to an annual equivalent investment cost (investment amount divided by a sum of discount factors) of 28 $ per ton of installed capacity.

The yields obtained are given in Table 1, which also shows the demand associated with each of the products and the operating costs.

Table 1

	"Topping reforming" scheme	"Topping reforming cracking" scheme	Demand to be satisfied (millions of tons)
Gasoline (Yield per ton of processed crude oil)	25%	40%	2
Diesel (Yield per ton of processed crude oil)	35%	35%	2
Heavy fuel oil (Yield per ton of processed crude oil)	35%	20%	1
Purchase and processing costs (dollar per ton of processed crude oil)	100	110	

The objective function (total cost) to be minimized can be formulated as follows:

- Operating cost associated with the purchase and processing of the crude oil:
  $F_1 = 100x + 110y$

- Annual equivalent investment cost representing the investment outlay:
  $F_2 = 28 \times 0.25y = 7y$

The objective function is thus broken down into two elementary functions.

To write the program in its simplicial form, we shall introduce slack variables, letting *e*, *g* and *f* denote (any) surpluses of finished products. By resuming the notations introduced:

$$X = \begin{bmatrix} x \\ y \\ e \\ g \\ f \end{bmatrix} \qquad A = \begin{bmatrix} 0.25 & 0.4 & -1 & 0 & 0 \\ 0.35 & 0.35 & 0 & -1 & 0 \\ 0.35 & 0.2 & 0 & 0 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$F = F_1 + F_2 = (100x + 110y) + 7y$$

Let Π_1, and Π_2 denote the vectors, formed of elementary dual values, respectively associated with the functions F_1, and F_2. As we have seen, it is possible to determine these two vectors in any feasible solution of the program. Here, we shall directly select the optimal solution so that the concept of long run marginal cost has meaning. At the optimum, the basic variables are *x, y* and *f*, and the null variables are *e* and *g*. We can write:

$$B = \begin{bmatrix} 0.25 & 0.4 & 0 \\ 0.35 & 0.35 & 0 \\ 0.35 & 0.2 & -1 \end{bmatrix} \text{ which gives: } B^{-1} = \begin{bmatrix} -6.67 & 7.62 & 0 \\ 6.67 & -4.76 & 0 \\ -1 & 1.71 & -1 \end{bmatrix} \text{ and } \overline{B} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

The basic variables have the value: $Y = B^{-1}b = \begin{bmatrix} 1.9 \\ 3.81 \\ 0.43 \end{bmatrix}$

The vectors formed of the elementary dual values are the following:

$$\Pi_1 = D_1 B^{-1} = [100\ \ 110\ \ 0] B^{-1} = [66.7\ \ 238.4\ \ 0]$$
$$\Pi_2 = D_2 B^{-1} = [0\ \ 7\ \ 0] B^{-1} = [46.7\ \ \text{-}33.3\ \ 0]$$

The vector Π (formed of the total dual values) gives the long run marginal cost for each of the three finished products:

$$\Pi = \Pi_1 + \Pi_2 = [113.4\ \ 205.1\ \ 0]$$

The expression of the basic variables as a function of the null (nonbasic) variables is also written (by calculating $B^{-1}\overline{B}$):

$$\begin{bmatrix} x \\ y \\ f \end{bmatrix} = \begin{bmatrix} 1.90 \\ 3.81 \\ 0.43 \end{bmatrix} + \begin{bmatrix} \text{-}6.67 & 7.62 \\ 6.67 & \text{-}4.76 \\ -1 & 1.71 \end{bmatrix} \begin{bmatrix} e \\ g \end{bmatrix}$$

We shall now interpret the dual values obtained. Firstly, the dual value associated with the heavy fuel demand constraint is null. This is explained by the fact that the slack variable associated with this constraint is a basic (non-null) variable at the optimum and does not appear in the objective function. This is a well known result: a non-null slack variable represents an unsaturated constraint, for which the associated dual value is null. The long-run marginal cost associated with the gasoline demand constraint (given by the first component of the vector Π) amounts to 113.4 \$ per ton. Let us look closely at the two terms making it up (given by the first component of the vector Π_1 and that of the vector Π_2). Increasing gasoline demand leads to the substitution of crude oil *y* for crude oil *x*, without any change in the total quantity of crude oil. The marginal operating cost amounts to 66.7 \$ per ton. This result is directly related to processing by the "topping reforming cracking" scheme of a larger quantity of crude oil. For the same reason, if an additional ton of gasoline were produced, a marginal equivalent investment cost of 46.7 \$ would have to be allocated to it (first component of the vector Π_2). The formation of the long run marginal cost associated with the gasoline is thus analyzed. A similar breakdown can be made for the long run marginal cost of diesel. This amounts to 205.1 \$. An increase in diesel demand, while giving rise to the processing of a larger total quantity of crude oil, also culminates in substituting crude oil *x* for crude oil *y*. Since the total quantity of processed crude oil increases, the marginal operating cost is positive (238.4 \$/t). Conversely, as the quantity of crude oil *y* processed decreases, the marginal equivalent investment cost is negative (-33.3 \$/t).

The marginal equivalent investment cost thus defined clearly has an average-cost structure:

$$\Pi_2 b = (46.7 \times 2) - (33.3 \times 2) + (0 \times 1.43) = 7 \times 3.81 = D_2 Y = F_2$$

This average-cost structure is also confirmed by the marginal operating cost:

$$\Pi_1 b = (66.7 \times 2) + (238.4 \times 2) + (0 \times 1.43) = F_1$$

4. Conclusion

In capital budgeting, the interpretation of long run marginal costs can sometimes prove to be difficult. The use of the theorem presented allows their breakdown into marginal operating costs and marginal equivalent investment cost. The introduction of taxation into the model raises no problem and only leads to a modification of the objective function. This theorem, with the associated concept of "elementary dual variable", should give rise to a sizable number of applications in capital budgeting modeling (for instance, in the presence of capital rationing constraints).

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