Stress-Induced Seismic Anisotropy Revisited

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STRESS-INDUCED SEISMIC ANISOTROPY REVISITED

P. RASOLOFOSAON

Institut français du pétrole[1]

NOUVEAU REGARD SUR L'ANISOTROPIE SISMIQUE INDUITE PAR LES CONTRAINTES

Un principe général esquissé par P. Curie (1894) concernant l'influence de la symétrie sur les phénomènes physiques dit, en langage actuel, que le groupe de symétrie des causes est un sous-groupe du groupe de symétrie des effets. Par exemple, en ce qui concerne l'anisotropie sismique induite par les contraintes, la symétrie la plus complexe présentée par un milieu initialement isotrope, sous contrainte triaxiale, est orthorhombique ou orthotrope, caractérisée par trois plans de symétrie orthogonaux deux à deux (Nur, 1971).

À d'autres égards, Schwartz et al. (1994) ont montré que deux modèles de roches très différents, un modèle fissuré et un modèle granulaire faiblement consolidé, conduisent toujours à une anisotropie elliptique quand ils sont soumis à une contrainte uniaxiale. La question posée est la suivante : est-ce que ce résultat est vrai pour tous les modèles de roches ? et, plus généralement, est-ce que les roches initialement isotropes, quand elles sont soumises à une contrainte triaxiale, forment un sous-ensemble bien défini des milieux orthorhombiques ?

Sous l'hypothèse d'hyperélasticité isotrope non linéaire du troisième ordre (c'est-à-dire absence d'hystérésis, et existence d'une fonction d'énergie élastique développée au troisième ordre dans les composantes de la déformation), il est démontré que l'anisotropie de l'onde qP induite par les contraintes est toujours ellipsoïdale, pour tout degré d'anisotropie. Par exemple, les sources ponctuelles engendrent des fronts d'onde qP de forme ellipsoïdale. Ce résultat est général et est absolument indépendant du modèle de roche, c'est-à-dire indépendant des causes de la non-linéarité, pour autant que les hypothèses de départ soient vérifiées. Ceci constitue le principal résultat de cet article.

Thurston (1965) a remarqué, vis-à-vis des propriétés élastiques, qu'un milieu élastique initialement isotrope, quand il est soumis à des contraintes non isotropes, n'est jamais tout à fait équivalent à un cristal anisotrope non soumis à des contraintes. Par exemple, les composantes du tenseur d'élasticité du milieu sous contrainte ne présentent pas la symétrie habituelle vis-à-vis de la permutation des indices. Ceci interdit l'emploi de la notation de Voigt sur les indices contractés. Toutefois, si l'amplitude des composantes du déviateur de contrainte est petite par rapport aux modules d'onde, ce qui est toujours vérifié sur le terrain en exploration sismique, l'équivalence parfaite est rétablie. Sous cette condition, les 9 rigidités élastiques C'_{IJ} (en notation contractée) d'un solide initialement isotrope, soumis à une contrainte triaxiale, sont toujours liées par les 3 conditions ci-après d'ellipticité dans les plans de coordonnées associés aux directions propres de la précontrainte statique.

(1) 1 et 4, avenue de Bois-Préau,
92852 Rueil-Malmaison Cedex - France

$$\begin{cases} \left(C'_{11} - C'_{66}\right)\left(C'_{22} - C'_{66}\right) - \left(C'_{12} + C'_{66}\right)^2 = 0 \; dans\, le\, plan\, xy \\ \left(C'_{11} - C'_{55}\right)\left(C'_{33} - C'_{55}\right) - \left(C'_{13} + C'_{55}\right)^2 = 0 \; dans\, le\, plan\, xz \\ \left(C'_{22} - C'_{44}\right)\left(C'_{33} - C'_{44}\right) - \left(C'_{23} + C'_{44}\right)^2 = 0 \; dans\, le\, plan\, yz \end{cases}$$

Ainsi, seulement 6 des 9 rigidités élastiques du solide contraint orthorhombique sont indépendantes (Nikitin et Chesnokov, 1981), et sont des fonctions simples des contraintes principales, des 2 constantes élastiques linéaires (du deuxième ordre) et des 3 constantes non linéaires (du troisième ordre) du solide isotrope non soumis à contrainte. De plus, à partir d'un état de pré-contrainte, le degré d'anisotropie de l'onde P ou S induite par la contrainte et la biréfringence de l'onde S (mais pas l'amplitude des modules d'onde eux-mêmes) sont déterminés par seulement 2 paramètres intrinsèques du milieu, un pour l'onde P et un pour les ondes S.

Les milieux élastiques isotropes soumis à une contrainte triaxiale constituent un sous-ensemble particulier des milieux orthorhombiques, appelés ici « milieux ellipsoïdaux », vérifiant les conditions énoncées ci-dessus. L'anisotropie ellipsoïdale est une généralisation naturelle de l'anisotropie elliptique. L'anisotropie ellipsoïdale est pour la symétrie orthorhombique ce qu'est l'anisotropie elliptique pour la symétrie transverse isotrope (TI). L'anisotropie elliptique est un cas particulier de l'anisotropie ellipsoïdale limitée aux milieux TI. En d'autres termes, l'anisotropie ellipsoïdale dégénère en une anisotropie elliptique dans les milieux TI. Dans les milieux ellipsoïdaux, la surface de lenteur de l'onde qP est toujours une ellipsoïde. Par contre, les surfaces de lenteur des ondes S ne sont pas ellipsoïdales, sauf dans le cas elliptique dégénéré, et doivent être considérées comme une surface unique entrecroisée et à valeur double (Helbig, 1994). Les intersections de ces surfaces avec les plans de coordonnées sont soit des ellipses, dans le cas où l'onde S est polarisée en dehors des plans coordonnés, soit des cercles, dans le cas où l'onde qS est polarisée dans le plan de coordonnées.

Le recueil très détaillé établi par Thomsen (1986) de données expérimentales sur l'anisotropie sismique des roches (considérées comme transversalement isotropes) montre que l'anisotropie elliptique constitue plus une exception qu'une règle. Étant donné que l'anisotropie induite par les contraintes est essentiellement elliptique quand elle est limitée aux milieux transversalement isotropes, cette étude montre donc clairement que la contrainte peut être pratiquement exclue en tant que cause directe unique de l'anisotropie élastique dans les roches.

STRESS-INDUCED SEISMIC ANISOTROPY REVISITED

A general principle outlined by P. Curie (1894) regarding the influence of symmetry in physical phenomena states, in modern language, that the symmetry group of the causes is a sub-group of the symmetry group of the effects. For instance, regarding stress-induced seismic anisotropy, the most complex symmetry exhibited by an initially isotropic medium when tri-axially stressed is orthorhombic, or orthotropic, symmetry characterized by three symmetry planes mutually perpendicular (Nur, 1971).

In other respects, Schwartz et al. (1994) demonstrated that two very different rock models, namely a cracked model and a weakly consolidated granular model, always lead to elliptical anisotropy when uniaxially stressed. The addressed questions are: Is this result true for any rock model? and more generally: Do initially isotropic rock form a well-defined sub-set of orthorhombic media when triaxially stressed?

Under the hypothesis of 3rd order nonlinear isotropic hyperelasticity (i.e., no hysteresis and existence of an elastic energy function developed to the 3rd order in the strain components) it is demonstrated that the qP-wave stress-induced anisotropy is always ellipsoidal, for any strength of anisotropy. For instance point sources generate ellipsoidal qP-wave fronts. This result is general and absolutely independent of the rock model, that is to say independent of the causes of nonlinearity, as far as the initial assumptions are verified. This constitutes the main result of this paper.

Thurston (1965) pointed out that an initially isotropic elastic medium, when non-isotropically pre-stressed, is never strictly equivalent to an unstressed anisotropic crystal. For instance the components of the stressed elastic tensor lack the familiar symmetry with respect to indices permutation. This would prohibit Voigt's notation of contracted indices. However if the magnitude of the components of the stress deviator is small compared to the wave moduli, which is always verified in practical situations of seismic exploration, the perfect equivalence is re-established. Under this condition, the 9 elastic stiffnesses C'_{IJ} (in contracted notation) of an initially isotropic solid, when triaxially stressed, are always linked by 3 ellipticity conditions in the coordinate planes associated with the eigen directions of the static pre-stress, namely:

$$\begin{cases} \left(C'_{11} - C'_{66}\right)\left(C'_{22} - C'_{66}\right) - \left(C'_{12} + C'_{66}\right)^2 = 0 \; in\ the\ xy - plane \\ \left(C'_{11} - C'_{55}\right)\left(C'_{33} - C'_{55}\right) - \left(C'_{13} + C'_{55}\right)^2 = 0 \; in\ the\ xz - plane \\ \left(C'_{22} - C'_{44}\right)\left(C'_{33} - C'_{44}\right) - \left(C'_{23} + C'_{44}\right)^2 = 0 \; in\ the\ yz - plane \end{cases}$$

Thus only 6 of the 9 elastic stiffnesses of the orthorhombic stressed solid are independent (Nikitin and Chesnokov, 1981), and are simple functions of the eigen stresses, and of the 2 linear (2nd order) and the 3 nonlinear (3rd order) elastic constants of the unstressed isotropic solid. Furthermore, given the state of pre-stress, the strength of the stress-induced P- or S-wave anisotropy and S-wave birefringence (but not the magnitude of the wave moduli themselves) are determined by only 2 intrinsic parameters of the medium, one for the P-wave and one for the S-waves.

Isotropic elastic media, when triaxially stressed, constitute a special sub-set of orthorhombic media, here called "ellipsoidal media", verifying the above conditions. Ellipsoidal anisotropy is the natural generalization of elliptical anisotropy. Ellipsoidal anisotropy is to orthorhombic symmetry what elliptical anisotropy is to transversely isotropic (TI) symmetry. Elliptical anisotropy is a special case of ellipsoidal anisotropy restricted to TI media. In other words, ellipsoidal anisotropy degenerates in elliptical anisotropy in TI media. In ellipsoidal media the qP-wave slowness surface is always an ellipsoid. The S-wave slowness surfaces are not ellipsoidal, except in the degenerate elliptical case, and have to be considered as a single double-valued self-intersecting sheet (Helbig, 1994). The intersections of these latter surfaces with the coordinate planes are either ellipses, for the S-vave polarized out of the coordinate planes, or circles, for the qS-wave polarized in the coordinate planes.

The nearly exhaustive collection of experimental data on seismic anisotropy in rocks (considered as transverse isotropic) by Thomsen (1986) show that elliptical anisotropy is more an exception than a rule. Since stress-induced anisotropy is essentially elliptical when restricted to transversely isotropic media, as a consequence this work clearly shows that stress can be practically excluded as a unique direct cause of elastic anisotropy in rocks.

REVISIÓN DE LA ANISOTROPÍA SÍSMICA INDUCIDA POR LOS ESFUERZOS

Un principio general destacado por P. Curie (1894) en relación con la influencia de la simetría sobre los fenómenos físicos establece, en un lenguaje moderno, que el grupo de simetría de las causas representa un subgrupo del grupo de simetría de los efectos. Por ejemplo, en relación a la anisotropía sísmica inducida por los esfuerzos la simetría más compleja que presenta un medio inicialmente isotrópico inicial cuando es sometido a un esfuerzo triaxial es la simetría ortorrómbica u ortotrópica, caracterizada por tres planos de simetría mutuamente perpendiculares (Nur, 1971).

En otros sentidos, Schwartz et al. (1994) demostraron que dos modelos de roca muy diferentes, concretamente un modelo con grietas y un modelo granular débilmente consolidado, conducen siempre a una anisotropía elíptica cuando son sometidos a un esfuerzo monoaxial. Las preguntas que se plantean son: ¿Es también cierto este resultado en caso de cualquier modelo de roca? y, más en general, ¿Forma la roca inicialmente isotrópica un subgrupo bien definido de medios ortorrómbicos cuando es sometida a un esfuerzo triaxial?

Bajo la hipótesis de una hiperelasticidad isotrópica no lineal de 3er orden (es decir, ausencia de histéresis y existencia de una función de energía elástica desarrollada hasta el 3er orden de los componentes de la deformacion), se demuestra que la anisotropía de onda qP inducida por los esfuerzos es siempre elipsoidal, para cualquiera potencia de anisotropía. Por ejemplo, las fuentes puntuales generan frentes de ondas qP elipsoidales. Este resultado es general y es absolutamente independiente del modelo de roca, es decir, independiente de las causas de no linealidad, siempre que las premisas iniciales sean verificadas. Esto constituye el resultado principal de este artículo.

Thurston (1965) indicó que un medio elástico inicialmente isotrópico, cuando es sometido a un esfuerzo no isotrópico, nunca es estrictamente equivalente a un medio anisotrópico no sometido a un esfuerzo. Por ejemplo, los componentes del tensor elástico de un medio sometido esfuerzos carecen de simetría de familia con respecto a la permutación de índices. Esto impedirá la notación de Voigt de índices concentrados. Sin embargo, si la magnitud del desviador del esfuerzo es pequeña en comparación con los módulos de onda, lo que siempre se verifica en situaciones prácticas de exploración sísmica, vuelve a restablecerse esta equivalencia perfecta. Bajo estas condiciones, los 9 mòdulos elásticos C'_{IJ} (en notación concentrada) de un sólido inicialmente isotrópico, cuando es sometido a esfuerzo triaxial, se encuentran siempre vinculadas con 3 condiciones de elipticidad en los planos de las coordenadas, asociadas con las direcciones principales del esfuerzo estàtico concretamente:

$$\begin{cases} \left(C'_{11} - C'_{66}\right)\left(C'_{22} - C'_{66}\right) - \left(C'_{12} + C'_{66}\right)^2 = 0 \; en\ el\ plano\ xy \\ \left(C'_{11} - C'_{55}\right)\left(C'_{33} - C'_{55}\right) - \left(C'_{13} + C'_{55}\right)^2 = 0 \; en\ el\ plano\ xz \\ \left(C'_{22} - C'_{44}\right)\left(C'_{33} - C'_{44}\right) - \left(C'_{23} + C'_{44}\right)^2 = 0 \; en\ el\ plano\ yz \end{cases}$$

Por lo tanto, sólo 6 de los 9 módulos elásticos del sólido sometido a esfuerzo ortorrómbico son independientes (Nitkin y Chesnokov, 1981) y son funciones simples de los esfuerzos principales, y de las 2 constantes elásticas lineales (2do orden) y de las 3 no lineales (3er orden) del sólido isotrópico no sometido a tensión de compresión. Además, dado el esfuerzo stático, la magnitud de la anisotropía de las ondas P o S inducida por el esfuerzo y la birrefringencia de la onda S (pero no la magnitud de los módulos de onda mismos) son determinados por sólo 2 parámetros intrínsecos del medio, uno para la onda P y uno para las ondas S.

Los medios elásticos isotrópicos, cuando son sometidos a esfuerzo triaxial, constituyen un subgrupo especial de medios ortorrómbicos, llamados aquí "medios elipsoidales", que verifican las condiciones antes mencionadas. La anisotropía elipsoidal es la generalización natural de la anisotropía elíptica. La anisotropía elipsoidal es a la simetría ortorrómbica lo que la anisotropía eliptica es a la simetría transversalmente isotrópica (TI). La anisotropía elíptica es un caso especial de anisotropía elipsoidal restringida a medios TI. En otras palabras, la anisotropía elipsoidal degenera en anisotropía elíptica en medios TI. En medios elipsoidales, la superficie de la lentitud de la onda qP es siempre un elipsoide. Las superficies de la lentitud de las ondas S no son elipsoidales, excepto en el caso degenerado eliptico, y deben ser consideradas como una capa única de doble valor autointersectada (Helbig, 1994). Las intersecciones de estas últimas superficies con los planos de las coordenadas son, ya sea elipsis para la onda S polarizada a partir de los planos de las coordenadas, o bien círculos, para la onda qS polarizada en los planos de las coordenadas.

La recolección casi exhaustiva de Thomsen (1986) de datos experimentales sobre la anisotropia sismica en rocas (consideradas transversalmente isotrópicas) muestra que la anisotropía elíptica es más bien la excepción que la regla. Dado que la anisotropía inducida por esfuerzo es esencialmente elíptica cuando se restringe a medios transversalmente isotrópicos, en consecuencia este trabajo muestra claramente que el esfuerzo puede prácticamente ser excluido como causa directa única de anisotropía elástica en rocas.

INTRODUCTION

The role of stress, whether it be past or present, is fundamental in major geological phenomena observable at the surface of the earth (e.g., earthquakes, volcanoes, uplift of mountains and subsidence of ocean basins). The stresses modify not only the geometries but also the physical properties of the geological formations (e.g., Turcotte and Schubert, 1982; Anderson, 1989).

Many authors have studied the effect of stress on the elastic properties of solids. Two classes of theory have been developped. In the first class, to quote Dahlen (1972): "*no assumptions (...) are made about the initial stress except that it satisfy the condition of equilibrium in the initial configuration*" (Dahlen, 1972; Kostrov and Nikitin, 1968; Nikitin and Chesnokov, 1981 and 1984). In the first of these references the magnitude of the stress was assumed small compared to the elastic moduli. Such an assumption was rejected in the other references. Using a perturbation theory Dahlen (1972) demonstrated that the stress-induced *P*-wave anisotropy is only of the second order in the ratio of the stress magnitude to the elastic moduli. However Nikitin and Chesnokov (1984) showed that this is true only if the unstressed medium is isotropic. Furthermore these authors show that an initially isotropic medium is necessarily of higher symmetry than orthorhombic when triaxially stressed. Note that this result can be simply deduced from Curie's principle (P. Curie, 1894; M. Curie, 1955; Shubnikov, 1988). Briefly said, this principle states that, for any physical phenomenon, the observable effects are at least as symmetric as their causes. In the case considered by Nikitin and Chesnokov there are 2 causes, namely the unstressed isotropic medium and the applied uniform static pre-stress. Since all these causes share at least 3 mutually perpendicular symmetry planes (i.e. the planes defined by any pair of eigendirections of stress), as a consequence the observable effects (i.e. any physical property of the medium, in particular the elastic properties) necessarily have these 3 symmetry planes. In other words the stressed medium is at least orthorhombic, but can be more symmetric. This is the case in fact, these authors also demonstrated that any isotropic medium, when tri-axially stressed, is not characterized by 9 independent elastic constants as any regular orthorhombic crystal (e.g. Auld, 1973) but by only 6 independent coefficients. Furthermore, taken in the context of seismic exploration the results of Dahlen (1974) show that the stress cannot be considered as a direct cause of anisotropy since the observable strength of anisotropy in the field are much larger than those expected from the theory.

In a second class of theories it is assumed that the state of pre-stress is achieved by reversible processes (Truesdell, 1961; Thurston and Brugger, 1964; Thurston, 1965). Johnson and Rasolofosaon (1996) used this type of theory to explicitly express the stress induced anisotropy in terms of the elastic coefficients (linear and nonlinear) of the medium, which can be easily extracted from data on rocks in the literature. They show that the uniaxial stress-induced effects (i.e. *P*-wave anisotropy and *S*-wave birefringence) always goes as $\sin^2\theta$, θ being the angle between the propagation direction and the stress direction. Closely related to that, Schwartz *et al.* (1994) show that two contrasted rock models, namely a cracked model and a granular model, both lead to elliptical anisotropy (Helbig, 1983) when uniaxially stressed.

All these results seem to indicate that isotropic media, when triaxially stressed, form a well-defined sub-set of orthorhombic media. The purpose of this paper is to demonstrate that this is true for any rock model, provided some reasonable assumptions, not very restrictive in practice. Some of the results presented here may have been already published but are presented here in a different unified way, and some other results are new. This work mainly represents an extension of the work of Johnson and Rasolofosaon (1996).

Because of the availability of data in the literature and because of the simple link with experimental data, we choose to follow the theory of Truesdell (1961) and Thurston and Brugger (1964). Their basic assumptions and the results in arbitrary anisotropic media are summarized in the first section, and specified for the case of an initially isotropic medium in the second section. Then we demonstrate in the next section that the stress-induced *P*-wave anisotropy is always "ellipsoidal". The results are much more complex for *S*-wave except in the case of propagation along the eigen directions of stress (4th section). A complicating factor for these waves comes from fact that a stressed isotropic medium is not stricly equivalent to an unstressed orthorhombic crystal, with respect to elastic properties. Fortunately, in practice, the deviations are so small that such subtelties can be neglected (5th section). Under this last assumption, in the next section we re-demonstrate that isotropic elastic media, when triaxially stressed, belong to a special class of orthorhombic (ORT) media, called "ellipsoidal" media in this work, characterized by 6 independent elastic constants, instead of 9 in conventional ORT media. This result was previously obtained by Nikitin and Chesnokov (1981) but in a substantially different way. In the following section we show that, given the state of pre-stress, the strength of the anisotropy (and not the wave moduli themselves) are essentially governed by 2 intrinsic parameters of the medium, one for the *P*-wave and the other for the *S*-waves. Then experimental data are presented and commented, followed by the conclusions.

1 WAVE-MODULI IN UNIFORMLY PRE-STRESSED ARBITRARY ANISOTROPIC ELASTIC SOLID

The first assumption is hyperelasticity, according to the terminology of Epstein and Slawinsky (this issue), which means that any hysteresis phenomena is excluded. In other words the existence of a strain energy function *E* is assumed. The second assumption is third order nonlinearity (NL) which means that the energy function *E* is developped to the third power in the components of the strain tensor *E*, which can be written for arbitrary anisotropic media:

$$E = \frac{1}{2} C_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl} + \frac{1}{6} C_{ijklmn}\, \varepsilon_{ij}\, \varepsilon_{kl}\, \varepsilon_{mn} \qquad (1)$$

where C_{ijkl} and C_{ijklmn} designate the components of the second order elastic (SOE) tensor and the third-order elastic (TOE) tensor, respectively. An arbitrary (triclinic) medium is characterized by 21 SOE independent constants and 56 TOE independent constants, whereas the most symmetric (isotropic) medium by only 2 SOE constants and 3 TOE constants (Brugger, 1965). The derivation of Equation (1) with respect to the components of the strain tensor *E* leads to the stress/strain relation, a modified Hooke's law, which injected in Newton's fundamental dynamic relation leads to the nonlinear elastodynamic equation. Within such a context many elastodynamic problems have been solved (e.g. Green, 1973). Of particular interest for our purpouse is the problem of elastic wave propagation in statically pre-stressed anisotropic media which has been solved by Thurston and Brugger (1964) and Thurston (1965). For instance, in the case of a uniaxial pre-stress, the stress derivative of the wave modulus of any bulk wave writes:

$$\left(\frac{\partial M}{\partial \sigma}\right)_{\sigma=0} = -(n.m)^2 - 2wF - G \qquad (2)$$

where

$$\left.\begin{aligned} w &= C_{ijkl}\, n_i\, n_k\, p_j\, p_l \\ F &= S_{ijkl}\, m_i\, m_j\, p_k\, p_l \\ G &= S_{ijkl}\, C_{kltrsq}\, m_i\, m_j\, n_t\, n_s\, p_r\, p_q \end{aligned}\right\} \qquad (3)$$

In Equation (2) *M* designates a wave modulus, that is to say the unstressed density multiplied by the "natural" velocity (i.e., the length of the "acoustical path" in the unstressed state divided by the wave travetime in the stressed state), and n_i and m_j are the components of the unit vectors **n** and **m** in the direction of wave propagation in the unstressed state and of the uniaxial stress, respectively. And in Equation (3) p_j and S_{ijkl} are the components of the polarization vector and of the compliance tensor, respectively, in the unstressed state. Note that, in contrast to the previous authors, we have not made any distinction between isothermal stiffness (or compliance) tensors and isentropic stiff-ness (or compliance) tensors normally appearing in the two previous equations. The difference is of the same order as or even smaller than the combined deviations due to errors of measurements of their components and the variability of the elastic properties in the material, as pointed out by Hearmon (1961).

If the uniform static pre-stress is triaxial, the wave modulus simply becomes (Johnson and Rasolofosaon, 1996):

$$M(\sigma) = M(\sigma=0) + \sigma_1 \left(\frac{\partial M}{\partial \sigma_1}\right)_{\sigma_1=0} + \sigma_2 \left(\frac{\partial M}{\partial \sigma_2}\right)_{\sigma_2=0} + \sigma_3 \left(\frac{\partial M}{\partial \sigma_3}\right)_{\sigma_3=0} \qquad (4)$$

where *M* (σ = 0) is the unstressed wave modulus. The stress tensor σ has been decomposed into its eigensystem $\sigma = (\sigma_1 (\mathbf{m}^{(1)}), \sigma_2 (\mathbf{m}^{(2)}), \sigma^3 (\mathbf{m}^{(3)}))$, where $\sigma_i (\mathbf{m}^{(i)})$ (i = 1, 2, 3) are mutually perpendicular uniaxial stresses in directions of the unit vectors $\mathbf{m}^{(i)}$. In this equation the partial derivatives of the wave modulus with respect to σ_i (i = 1, 2, 3) are given by Equation (2) in which one lets $\sigma = \sigma_i$ (i, 1, 2, 3). The reference axes XYZ of description of the problem are aligned with the eigen directions of stress.

2 SPECIAL CASE OF MEDIA INITIALLY ISOTROPIC WITH RESPECT TO BOTH LINEAR AND NONLINEAR ELASTIC PROPERTIES

In addition to the aforementioned assumptions, one assumes that the medium can be considered as isotropic with respect to both linear and NL elastic properties. The strain energy E of Equation (1) simplifies:

$$\begin{aligned} E &= \frac{1}{2}\left(K+\frac{4}{3}\mu\right)\varepsilon_{ii}\varepsilon_{kk} - \mu(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ij}) \\ &+ \frac{1}{3}(l+2m)\varepsilon_{ii}\varepsilon_{jj}\varepsilon_{kk} - m\varepsilon_{kk}(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ij}) + n\det(\varepsilon) \end{aligned} \qquad (5)$$

where det (ε) is the determinant of the strain matrix. K and μ are the SOE constants (respectively bulk and shear moduli), and, l, m and n the TOE constants, or Murnaghan coefficients (Murnaghan, 1951). The 3rd order NL properties can be characterized by other sets of 3 TOE constants. Except Murnaghan's set of constants the most frequently used sets of TOE constants in the literature are the crystallographic set (C_{111}, C_{112}, C_{123}) (Brugger, 1965 and Landau's set A, B, C; Landau and Lifschitz, 1959). All these sets of TOE constants are linked by the following relations (Green, 1973):

$$\left.\begin{aligned} C_{111} &= 2A + 6B + 2C = 2l + 4m \\ C_{112} &= 2B + 2C = 2l \\ C_{123} &= 2C = 2l - 2m + n \end{aligned}\right\} \qquad (6)$$

Under uniaxial pre-stress the stress derivative of the wave moduli (Eq. (2)) takes the simplified form (Johnson and Rasolofosaon, 1996):

$$\left(\frac{\partial M^{(K)}}{\partial \sigma}\right)_{\sigma=0} = C^{(K)}\cos^2\theta + S^{(K)}\sin^2\theta \qquad (7)$$

where θ is the angle between the stress direction **m** and the propagation direction **n**, and (K) designates the wave type. One lets (K) = (P) for a P-wave, (K) = (in-plane) for a S-wave polarized in the plane defined by **m** and **n**, and (K) = (out-plane) for an S-wave polarized out of this plane. The constants $C^{(K)}$ and $S^{(K)}$ of Equation (7) have the following expressions:

for a P-wave:

$$\left.\begin{aligned} C^{(P)} &= -1 - 2C_{33}S_{11} - \left[C_{111}S_{11} + 2C_{112}S_{12}\right] \\ S^{(P)} &= -2C_{33}S_{12} - \left[C_{111}S_{12} + C_{112}\left(S_{11} + S_{12}\right)\right] \end{aligned}\right\} \qquad (8)$$

for an S-wave polarized in the plane defined by **m** and **n**:

$$\left.\begin{aligned} C^{(in\text{-}plane)} &= C^{(S)} = -1 - 2C_{44}S_{12} \\ &\quad - \frac{1}{4}\left[C_{111}\left(S_{11}+S_{12}\right) + C_{112}\left(S_{12}-S_{11}\right) - 2C_{123}S_{12}\right] \\ S^{(in\text{-}plane)} &= -2C_{44}S_{11} \\ &\quad - \frac{1}{4}\left[C_{111}\left(S_{11}+S_{12}\right) + C_{112}\left(S_{12}-S_{11}\right) - 2C_{123}S_{12}\right] \end{aligned}\right\} \qquad (9)$$

and for a S-wave of which the polarization vector is not contained in the plane (**n**, **m**):

$$\left.\begin{aligned} C^{(out\text{-}plane)} &= C^{(S)} = -1 - 2C_{44}S_{12} \\ &\quad - \frac{1}{4}\left[C_{111}\left(S_{11}+S_{12}\right) + C_{112}\left(S_{12}-S_{11}\right) - 2C_{123}S_{12}\right] \\ S^{(out\text{-}plane)} &= -2C_{44}S_{12} \\ &\quad - \frac{1}{2}\left[C_{111}S_{12} + C_{112}\left(S_{11}-S_{12}\right) - C_{123}S_{11}\right] \end{aligned}\right\} \qquad (10)$$

In Equations (8) to (10) one has:

$$C_{33} = K + \frac{4}{3}\mu;\; C_{44} = \mu;\; S_{11} = \frac{C_{33} + C_{12}}{\left(C_{33} - C_{12}\right)\left(C_{33} + 2C_{12}\right)}$$

where $C_{12} = C_{33} - 2C_{44}$; and $S_{12} = S_{11} - \dfrac{1}{2C_{44}}$.

Note that $C^{(in\text{-}plane)} = C^{(out\text{-}plane)}$, their common value being designated by $C^{(S)}$. Note also the slight modifications in the detailed expressions of Equation (7) compared with the corresponding expressions in the latter reference due to misprints.

3 P-WAVE STRESS-INDUCED ANISOTROPY IS ALWAYS ELLIPSOIDAL (IN INITIALLY ISOTROPIC MEDIA)

After Equations (4), (7) and (8) one can deduce the P-wave modulus in the presence of a triaxial pre-stress:

$$M^{(P)}(\sigma)=C_{33}+\left(\sigma_1+\sigma_2+\sigma_3\right)S^{(P)} + \left[C^{(P)}-C^{(S)}\right]\left(\sigma_1\cos^2\theta_1+\sigma_2\cos^2\theta_2+\sigma_3\cos^2\theta_3\right) \quad (11)$$

Because $\cos\theta_i = n_i$ are the direction cosines of the unit propagation vector $\mathbf{n}$, one has the relation $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = 1$. Multiplying the two first terms of the right-hand member of Equation (11) by $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3$ do not change this equation but transforms the expression of the P-wave modulus in the following way:

$$\begin{aligned} M^{(P)}(\sigma) = {} & \cos^2\theta_1\left\{\begin{aligned} & C_{33} + \left(\sigma_1+\sigma_2+\sigma_3\right)S^{(P)} \\ & + \sigma_1\left[C^{(P)} - S^{(P)}\right]\end{aligned}\right\} \\ & + \cos^2\theta_2\left\{\begin{aligned} & C_{33} + \left(\sigma_1+\sigma_2+\sigma_3\right)S^{(P)} \\ & + \sigma_2\left[C^{(P)} - S^{(P)}\right]\end{aligned}\right\} \\ & + \cos^2\theta_3\left\{\begin{aligned} & C_{33} + \left(\sigma_1+\sigma_2+\sigma_3\right)S^{(P)} \\ & + \sigma_3\left[C^{(P)} - S^{(P)}\right]\end{aligned}\right\} \end{aligned} \quad (12)$$

In this equation one lets:

$$X = \frac{\cos\theta_1}{\sqrt{M^{(P)}}}, \quad Y = \frac{\cos\theta_2}{\sqrt{M^{(P)}}} \quad \text{and} \quad Z = \frac{\cos\theta_3}{\sqrt{M^{(P)}}} \quad (13)$$

and

$$\left.\begin{aligned} a^{-2} &= C_{33} + \left(\sigma_1+\sigma_2+\sigma_3\right)S^{(P)} + \sigma_1\left[C^{(P)} - S^{(P)}\right] \\ b^{-2} &= C_{33} + \left(\sigma_1+\sigma_2+\sigma_3\right)S^{(P)} + \sigma_2\left[C^{(P)} - S^{(P)}\right] \\ c^{-2} &= C_{33} + \left(\sigma_1+\sigma_2+\sigma_3\right)S^{(P)} + \sigma_3\left[C^{(P)} - S^{(P)}\right] \end{aligned}\right\} \quad (14)$$

Note that X, Y and Z are the components of the P-wave slowness vector (provided that the P-wave modulus $M^{(P)}$ is density normalized), and a, b and c are the 3 principal P-wave slownesses along the eigen directions of stress. We have to remember that, after Equations (12) and (14), if the P-wave modulus $M^{(P)}$ is density normalized, the coefficients a^{-2}, b^{-2} *and* c^{-2} also are. Using Equations (13) and (14), Equation (12) can now be identified with the canonical equation of an ellipsoid:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} = 1 \quad (15)$$

Equations (13) to (15) simply mean that the P-wave slowness surface is always an ellipsoid in isotropic media which are triaxially stressed. This constitutes the main result of this paper. This result surprisingly seems to have escaped the previous authors, as far as I am aware. It is worth noting that, up to now, the only assumptions are third order nonlinear hyperelasticity and perfect isotropy (with respect to both linear and NL elastic properties) of the unstressed medium. But the anisotropy induced by the applied stress can be of any magnitude. All the results obtained so far are general and absolutely independent of the rock model, that is to say independent of the causes of the nonlinearity as far as the aforementioned assumptions are verified.

4 S-WAVES MODULI ALONG THE COORDINATE AXES

In the case of propagation along the coordinate axes, the polarizations of both shear waves are perfectly defined and the expressions of the moduli of both S-waves, similar to Equation (11) for the P-wave, can be straightforwardly deduced from Equations (4), (7), (9) and (10). Letting M_{IJ} the modulus of the wave propagating along the I-axis and polarized along the J-axis, one has:

$$\left.\begin{aligned} M_{XY} &= C_{44} + C^{(S)}\sigma_1 + S^{(in\text{-}plane)}\sigma_2 + S^{(out\text{-}plane)}\sigma_3 \\ M_{XZ} &= C_{44} + C^{(S)}\sigma_1 + S^{(out\text{-}plane)}\sigma_2 + S^{(in\text{-}plane)}\sigma_3 \end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned} M_{YX} &= C_{44} + S^{(in\text{-}plane)}\sigma_1 + C^{(S)}\sigma_2 + S^{(out\text{-}plane)}\sigma_3 \\ M_{YZ} &= C_{44} + S^{(out\text{-}plane)}\sigma_1 + C^{(S)}\sigma_2 + S^{(in\text{-}plane)}\sigma_3 \end{aligned}\right\} \quad (17)$$

and:

$$\left.\begin{aligned} M_{ZX} &= C_{44} + S^{(in\text{-}plane)}\sigma_1 + S^{(out\text{-}plane)}\sigma_2 + C^{(S)}\sigma_3 \\ M_{ZY} &= C_{44} + S^{(out\text{-}plane)}\sigma_1 + S^{(in\text{-}plane)}\sigma_2 + C^{(S)}\sigma_3 \end{aligned}\right\} \quad (18)$$

Outside the coordinate planes the procedure used for the P-wave cannot be directly applied to the S-waves. More precisely, because of the well-known phenomenon of interchange of polarizations between the 2 shear waves (Helbig, 1994), it is not straightforward to apply Equations (4), (7), (9) and (10) to the S-waves in arbitrary directions of propagation. Nevertheless we shall see in the next section that, provided an additional but not very restrictive assumption holds, one can go much further in the description of the results and in the derivation of the S-wave moduli. But before that we first recall some important differences between stress-induced anisotropy and anisotropy in the absence of stress.

5 DIFFERENCES BETWEEN STRESS-INDUCED ANISOTROPY AND ANISOTROPY IN ABSENCE OF STRESS

As pointed out by Thurston (1965), the elastic constants C'_{ijkl} (with non-contracted indices) of a stressed elastic medium lack the familiar symmetry with respect to permutations of indices conventionally encountered in unstressed anisotropic media, namely $C'_{ijkl} = C'_{jikl} = C'_{ijlk} = C'_{klij}$. For instance, the wave modulus of the S-wave propagating along the z-axis and polarized along the x-axis is by definition C'_{1313}, and is equal to M_{zx} after Equation (18). In contrast the S-wave for which these directions of propagation and polarization are interchanged has, by definition, a modulus equal to C'_{3131}, or to M_{xz} after Equation (16). One clearly sees from Equations (16) and (18) that $C'_{1313} = M_{zx}$ differs from $C'_{3131} = M_{xz}$ by a quantity equal to the difference $\sigma_1 – \sigma_3$ between the principal stresses in the X and Z directions. This lack of familiar symmetry with respect to permutation of the indices of the elastic coefficients constitutes a notable difference between stress-induced anisotropy and conventional anisotropy. In particular this implies, firstly, that Voigt's notation is not applicable in stressed anisotropic media and, secondly, that a stressed anisotropic medium is in general not mechanically equivalent to an unstressed anisotropic crystal (Thurston, 1965), except under an isotropic state of stress (hydrostatic stress). In this last case the familiar symmetry of the indices still holds.

In spite of these differences the effect is small and can be neglected in practical situations as noted by Thurston (1965). More precisely, the wave moduli in rocks typically are of the order of 10^{10} Pa. In contrast the wave moduli differences which impede the familiar symmetry of the indices are equal to the magnitude of the components of the stress deviator, which can hardly exceed a few 10^7 Pa at typical depths of oil reservoirs. This implies that neglecting all these subtleties leads to relative errors on the wave moduli of the order of 0.1% (the ratio between the magnitude of the stress deviator components and the wave moduli), and of the order of 0.05% on the velocities. These errors are negligible in all practical situations of seismic exploration. Thus the assumption that a stressed initially isotropic medium is equivalent to an unstressed orthorhombic medium, which appeared as a naive over-simplification in the context of a strict nonlinear elastic theory, is proved to be correct in practice. This leads to important relations between the elastic constants of the stressed medium which will be developed in the next section.

6 RELATIONS BETWEEN THE ELASTIC CONSTANTS OF PRE-STRESSED ISOTROPIC MEDIA - ELLIPSOIDAL MEDIA

As previously mentioned, under the hypotheses of 3rd order nonlinear (NL) hyperelasticity (i.e. no hysteresis and existence of a strain energy function) and perfect isotropy (with respect to both linear and NL properties) of the unstressed medium, and provided that the magnitude of the components of the stress deviator is small compared to the wave moduli, the stressed elastic medium can be described as a conventional orthorhombic medium (after Curie's principle). Since the P-wave slowness surface is always an ellipsoid, then the section of this surface by the coordinate planes are ellipses. Thus 3 ellipticity conditions must be verified (Helbig, 1994) by the stressed elastic constants C'_{IJ} (now in Voigt's contracted indices notation), namely:

$$\left.\begin{aligned}
&\left(C'_{11}-C'_{66}\right)\left(C'_{22}-C'_{66}\right)-\left(C'_{12}+C'_{66}\right)^2=0 \quad \textit{in the xy-plane}\\
&\left(C'_{11}-C'_{55}\right)\left(C'_{33}-C'_{55}\right)-\left(C'_{13}+C'_{55}\right)^2=0 \quad \textit{in the xz-plane}\\
&\left(C'_{22}-C'_{44}\right)\left(C'_{33}-C'_{44}\right)-\left(C'_{23}+C'_{44}\right)^2=0 \quad \textit{in the yz-plane}
\end{aligned}\right\} \quad (19)$$

Equation (19) implies that initially isotropic media, when triaxially stressed, form a well-defined sub-set of orthorhombic media which are not characterized by 9 independent elastic coefficients, as conventional orthorhombic media, but by only 6 independent coefficients. This corroborates a result previously derived by Nikitin and Chesnokov (1981). Yet the 3 relations derived by these authors have notably different forms and do not seem to have simple physical interpretation.

We shall call the particular class of orthorhombic media verifying the conditions (19) "ellipsoidal" media. In ellipsoidal media the *P*-wave slowness surface, and as a consequence the *P*-wave surface (because these surfaces are reciprocal), are always strictly ellipsoidal. For instance point sources always generate ellipsoidal *P* wave-fronts in such media. Ellipsoidal anisotropy is to orthorhombic symmetry what the more conventional elliptical anisotropy is to TI symmetry. Thus elliptical anisotropy is a particular case of ellipsoidal anisotropy restricted to the classes of TI media.

Regarding the *S*-waves, things are more complicated in ellipsoidal media than in elliptical media. Let us remember that in elliptical TI media the *S*-wave polarized in the sagittal plane (i.e. the plane defined by the TI axis and by the propagation direction) is isotropic, and the other *S*-wave is elliptical as in any TI medium (Helbig, 1983). In contrast in ellipsoidal orthorhombic media the shear wave slowness surfaces have to be considered as a single double-valued self intersecting sheet (Helbig, 1984). In such media the *S*-wave slowness surfaces are never ellipsoidal, in contrast with *P*-wave slownesses, except of course when ellipsoidal anisotropy degenerates in elliptical anisotropy. The reader can refer to Helbig's textbook to find more detailed considerations about ellipsoidal media. Note that the previous author do not use this terminology which we found to be the natural generalization of the conventional term "elliptical".

We mentioned that only 6 independent elastic constants (instead of 9 in conventional orthorhombic media) defines an ellipsoidal medium. These constants can be chosen as the diagonal elements of the elasticity matrix. In orthorhombic media, and in particular in ellipsoidal media, these elastic constants correspond to the moduli *P*- or *S*-waves propagating and polarized along the coordinate axes (Auld, 1973). For the *P*-waves their expressions are easily deduced from Equations (11) and (12) and for *S*-waves from Equations (16) to (18). Given these constants, the 3 remaining off-diagonal elastic constants, namely C'_{12}, C'_{23} and C'_{13} are determined by the relations (19). Finally the general explicit expressions of the moduli of the 3 bulk waves are identical to the corresponding expressions in orthorhombic media (Helbig, 1984) and need not be reported here.

7 STRESS-INDUCED ANISOTROPY AND BIREFRINGENCE COEFFICIENTS

In this section we are more concerned with the strength of the anisotropy and of the birefringence due to the applied stress than with the absolute values of the wave moduli. In other words, given the static pre-stress (i.e. eigen directions and eigenvalues) we address the following question: How many intrinsic parameters are necessary to quantify the strength of the anisotropy induced by a static pre-stress? We mean by intrinsic parameters the parameters characterizing the considered medium itself, independently of the geometry of the problem and of external factors such as the stress. We shall demonstrate that only 2 parameters intrinsically characterize the strength of the anisotropy due to the applied stress.

It was mentioned above that the complete directional dependence of the 3 bulk waves are characterized by only 6 independent coefficients. As explained before these 6 coefficients can be chosen as the diagonal elements of the stiffness matrix, which correspond to the moduli of *P*- and *S*-waves propagating and polarized along the eigen directions of stress. In such media the *P*- and *S*-wave anisotropy in the coordinate planes perfectly determines their anisotropy outside these planes because of the 3 ellipticity conditions (Eq. (19)). (This is always the case except in the "anomalous orthorhombic anisotropy" discussed by Carcione and Helbig during the *7IWSA* in Miami).

In order to quantify the strength of anisotropy we use the dimensionless anisotropy parameters introduced by Mensch and Rasolofosaon (1997) and generalizing Thomsen's parameters ε and γ (Thomsen, 1986). For orthorhombic media the former authors introduced the parameters ε_i and γ_i ($i = x, y, z$) to quantify the anisotropy in the plane normal to the *i*-axis, of the *P*-wave and of the *S*-wave polarized along the *i*-axis. But contrary to these authors, and in order not to privilege any direction of propagation, the normalization coefficients of these anisotropy parameters will be kept constant and equal to C_{33} and C_{44} for P- and S-wave anisotropy, respectively. Thus the P-wave stress-induced anisotropy is equal to:

$$\left.\begin{aligned}
\varepsilon_z &= \frac{M_{XX} - M_{YY}}{2C_{33}} \\
&= \frac{C^{(P)} - S^{(P)}}{2C_{33}}(\sigma_1 - \sigma_2) \quad \textit{in the xy-plane} \\
\varepsilon_x &= \frac{M_{YY} - M_{ZZ}}{2C_{33}} \\
&= \frac{C^{(P)} - S^{(P)}}{2C_{33}}(\sigma_2 - \sigma_3) \quad \textit{in the yz-plane} \\
\varepsilon_y &= \frac{M_{ZZ} - M_{XX}}{2C_{33}} \\
&= \frac{C^{(P)} - S^{(P)}}{2C_{33}}(\sigma_3 - \sigma_1) \quad \textit{in the zx-plane}
\end{aligned}\right\} \tag{20}$$

where M_{XX}, M_{YY} and M_{ZZ} are the P-wave moduli along the coordinate axes, that is to say $M^{(P)}$ in Equation (11) with $(\theta_1, \theta_2, \theta_3) = (0°, 90°, 90°)$, $(\theta_1, \theta_2, \theta_3) = (90°, 0°, 90°)$ and $(\theta_1, \theta_2, \theta_3) = (90°, 90°, 0°)$, respectively. After this equation the stress-induced P-anisotropy in a coordinate plane is proportional to the difference between the eigenstresses in the considered plane multiplied by a constant coefficient A_P equal to:

$$A_P = \frac{C^{(P)} - S^{(P)}}{2C_{33}} = -\frac{K + \frac{4}{3}\mu + 2m}{2\mu\left(K + \frac{4}{3}\mu\right)} \tag{21}$$

This coefficient, introduced by Johnson and Rasolofosaon (1996), is the stress-induced P-wave anisotropy coefficient, and represents the P-wave anisotropy induced per unit of eigenstress difference in the plane of two eigenstresses. Note the slight differences with the previous reference due to typos.

Similarly one can introduce the stress induced anisotropy of the S-wave polarized out of the considered plane:

$$\left.\begin{aligned}
\gamma_z &= \frac{M_{XZ} - M_{YZ}}{2C_{44}} \\
&= \frac{C^{(S)} - S^{(out\text{-}plane)}}{2C_{44}}(\sigma_1 - \sigma_2) \quad \textit{in the xy-plane} \\
\gamma_x &= \frac{M_{YX} - M_{ZX}}{2C_{44}} \\
&= \frac{C^{(S)} - S^{(out\text{-}plane)}}{2C_{44}}(\sigma_2 - \sigma_3) \quad \textit{in the yz-plane} \\
\gamma_y &= \frac{M_{ZY} - M_{XY}}{2C_{44}} \\
&= \frac{C^{(S)} - S^{(out\text{-}plane)}}{2C_{44}}(\sigma_3 - \sigma_1) \quad \textit{in the xz-plane}
\end{aligned}\right\} \tag{22}$$

As for the P-wave, the S-wave stress-induced anisotropy in a coordinate plane is proportional to he difference between the eigenstresses in the considered plane multiplied by a constant coefficient B_S equal to:

$$B_S = \frac{C^{(S)} - S^{(out\text{-}plane)}}{2C_{44}} = -\frac{4\mu + n}{8\mu^2} \tag{23}$$

Instead of the S-wave anisotropy one can also introduces the S-wave birefringence b_i along the i–axis, which is equal to the normalized difference between the moduli of the two S-waves propagating along this direction:

$$\left.\begin{aligned}
b_z &= \frac{M_{ZX} - M_{ZY}}{2C_{44}} \approx \frac{M_{XZ} - M_{YZ}}{2C_{44}} = \gamma_z \\
&= \frac{C^{(S)} - S^{(out\text{-}plane)}}{2C_{44}}(\sigma_1 - \sigma_2) \quad \textit{along the z-axis} \\
b_x &= \frac{M_{XY} - M_{XZ}}{2C_{44}} \approx \frac{M_{YX} - M_{ZX}}{2C_{44}} = \gamma_x \\
&= \frac{C^{(S)} - S^{(out\text{-}plane)}}{2C_{44}}(\sigma_2 - \sigma_3) \quad \textit{along the x-axis} \\
b_y &= \frac{M_{YZ} - M_{YX}}{2C_{44}} \approx \frac{M_{ZY} - M_{XY}}{2C_{44}} = \gamma_y \\
&= \frac{C^{(S)} - S^{(out\text{-}plane)}}{2C_{44}}(\sigma_3 - \sigma_1) \quad \textit{along the y-axis}
\end{aligned}\right\} \tag{24}$$

Here we use the result $M_{IJ} \approx M_{JI}$, a direct consequence of the assumption that the magnitude of the components of the stress deviator is small compared to the magnitude of the wave moduli (see previous comments). One can notice that, as for the *S*-wave anisotropy (Eq. (22)), the stress-induced *S*-wave birefringence along any coordinate axis is proportional to the difference between the eigenstresses in the plane perpendicular to this axis multiplied by the coefficient B_S defined by Equation (23).

In conclusion, given the state of pre-stress, the stress-induced *P*- and *S*-wave anisotropy in any coordinate plane and the stress-induced *S*-wave birefringence along any coordinate axis are perfectly determined by the knowledge of only two intrinsic coefficients, namely A_P and B_S. Because of the 3 ellipticity conditions (Eq. (19)) all these quantities are also perfectly determined outside coordinate planes or axes.

8 EXPERIMENTAL RESULTS AND DISCUSSIONS

In order to illustrate the results of the previous sections we have summarized on Table 1 experimental data on rocks and on “homogeneous” reference materials for comparison. The considered rocks are cristalline rocks (Lujavarite, Pegmatite, Urtite, Juvite, Apatite, Urtite, Rischorrite, Ijolite and Apane) of Bakulin and Protosenya (1982), Barre granite, Fontainebleau sandstone F32 “natural” and “thermally cracked”, both dry sand water saturated, dry marble D82 (data listed by Johnson and Rasolofosaon, 1996), Pfalzian sandstone, Magnesian marble and Colorado oil shale (data from Rasolofosaon and Yin, 1996). There are few references in the literature that provide the complete set of TOE and SOE coefficients in rocks, as far as we are aware these are the only data available. The reference materials are Armco iron, polystyrene, and Pyrex glass (data from Hughes and Kelly, 1953), and two crystals, namely quartz and calcite considered as isotropic by Rasolofosaon and Yin, (1996).

The first column of Table 1 give the references in the literature and the material. The two following columns contain the 2 linear SOE constants (i.e. bulk modulus K and shear modulus μ). The three following columns list the 3 nonlinear TOE constants (i.e. the Murnaghan coefficients l, m and n). The two last columns list the 2 stress-induced anisotropy coefficients, namely A_P and B_S defined by Equations (21) and (23). Only the two last columns are obtained from this work, and the other ones are taken from the references and reported here for completeness and convenience.

Many of the results commented by Johnson and Rasolofosaon (1996) are corroborated by Table 1. The most important one is that, contrary to “intact” (crack-free) homogeneous solids, such as iron, glass or quartz and calcite crystals, rocks can exhibit strong nonlinearity, directly correlated with large stress induced anisotropy or birefringence. For instance, the magnitudes of the coefficients A_P and B_S are orders of magnitude larger in rocks than in intact materials. More detailed comments can be found in the previous reference.

Note that some of the rock samples studied by Bakulin and Protosenya (1982) exhibit much stronger nonlinearity and stress-induced effects than the other rocks (e.g., note the coefficients A_P and B_S of Juvite, Lujavarite # 2 and Rischorrite are larger by one to two orders of magnitude than those of the other rocks). As no detailed description of the samples is provided by these authors, we have no explanation of these observations.

Finally note that the coefficients A_P and B_S of the rocks considered by Johnson and Rasolofosaon (1996) are all positive. This means that the stress-induced effects on these samples are of the same type. For instance, in the presence of a uniaxial stress and for propagation in a direction perpendicular to the direction of stress, the fast *S*-wave and the slow *S*-wave are always polarized parallel and perpendicular to the applied stress, respectively. Regarding *P*-wave anisotropy in uniaxially stressed rocks, the *P*-wave propagating in the direction of the applied stress always propagate faster than the one propagating perpendicular to that direction. The rock samples studied by the other authors exhibit either similar behaviour (such as Pegmatite, Lujavarite # 2, Rischorrite and Ijolite), either completely opposite behaviour (such as Lujavarite # 1, Urtite # 1, Apatite # 1 and # 2, Juvite and Colorado oil shale), or “mixed” behaviour (for the other rocks) that is to say A_P and B_S of opposite sign.

TABLE 1

Second- and third-order elastic moduli and parameters characterizing stress-induced anisotropy in rocks compared to standard materials and crystals (references: [1] Hughes and Kelly (1953); [2] Bakulin and Protosenya (1982); [3] Johnson and Rasolofosaon (1996); [4] Rasolofosaon and Yin (1996)).

	Material	K **(GPa)**	m **(Gpa)**	l **(GPa)**	m **(GPa)**	n **(GPa)**	A_p **(Gpa^{-1})**	B_s **(GPa^{-1})**
[1]	Armco Iron	164.0	82.0	-348	-1030	1100	0.04	-0.03
[1]	Polystyrene	3.8	1.4	-19	-13	-10	1.19	0.28
[1]	Pyrex glass	31.8	27.5	14	92	420	-0.07	-0.09
[4]	Quartz	38.0	48.0	-98	-89	-165	0.0029	-0.0015
[4]	Calcite	76.0	37.0	-77	-136	-141	0.012	-0.0006
[3]	Barre	13.8	18.2	-3371	-6742	-6600	9.69	2.46
[3]	F32 dry	15.3	11.7	-97800	-99400	-84900	274.88	77.48
[3]	F32 cracked	7.8	5.7	-74000	-64500	-34900	734.67	134.18
[3]	F32 saturated	30.2	17.5	-59100	-38200	-27500	40.74	11.20
[3]	D82 dry	30.0	21.3	-40300	-35400	-20300	28.43	5.57
[4]	Pfalzian sandstone	7.3	6.3	-2530	-580	1140.	5.75	-3.67
[4]	Magnesian marble	15.0	14.0	-25150	-7090	13590	14.99	-8.70
[4]	Colorado oil shale	21.0	18.0	-66600	7230	24420	-8.96	-9.45
[2]	Lujavarite # 1	107.0	32.0	-510000	110000	79000	-22.99	-9.66
[2]	Pegmatite	58.5	32.9	-29500000	-5000000	-8100000	1484.6	935.40
[2]	Urtite # 1	71.2	18.8	-3250000	500000	5900000	-276.30	-2086.70
[2]	Juvite	86.6	24.4	-1800000	9300000	9300000	-2580.1	-1952.6
[2]	Apatite # 1	57.0	29.8	-286000	68000	31000	-23.61	-4.38
[2]	Lujavarite # 2	39.4	22.9	-20700000	-3400000	-5100000	2123.00	1215.60
[2]	Urtite # 2	64.1	25.4	-3510000	6000000	-282000	-2411.30	54.62
[2]	Rischorrite	57.1	28.1	-33600000	6000000	-6400000	2257.90	1013.10
[2]	Ijolite	55.6	27.8	-3080000	-330000	-560000	128.08	90.56
[2]	Apatite # 2	60.4	31.5	302000	51000	85000	-15.83	-10.72

We know that, at least for the data of Rasolofosaon and Yin (1996), this may be due to anisotropy, which is not taken into account in the last part of the present paper. In fact the rocks of this last reference cannot be considered as isotropic, especially with respect to nonlinear elastic properties, and the tabulated parameters are simply those of an "equivalent isotropic" medium. As a consequence, the tabulated A_p and B_s simply quantify some parameter averaged in all the directions of space, which can be either negative or positive.

CONCLUSIONS

Under the assumptions of 3rd order nonlinear hyperelasticity (i.e. no hysteresis and existence of an elastic energy function developed to the 3rd order in the strain components) and of isotropy with respect to both linear and nonlinear elastic properties, in addition to previously published results (references quoted below between brackets), it is demonstrated that:

– Initially isotropic media, when tri-axially stressed, are at least of orthorhombic symmetry (Nur, 1971;

Nikitin and Chesnokov, 1981 and 1984), a result which can simply be deduced from Curie's principle (P. Curie, 1894; M. Curie, 1955; Shubnikov, 1988).

– Initially isotropic media, when triaxially stressed, form a well-defined sub-set of orthorhombic media. The *qP*-wave stress-induced anisotropy is always ellipsoidal, for any strength of anisotropy. For instance point sources generate ellipsoidal *qP*-wave fronts. This result is general and absolutely independent of the rock model, that is to say independent of the causes of nonlinearity, as far as the initial assumptions are verified. This constitutes the main result of this paper.
– An initially isotropic elastic medium, when non-isotropically pre-stressed, is never strictly equivalent to an unstressed anisotropic crystal, with respect to elastic properties. For instance the components of the stressed elastic tensor lack the familiar symmetry with respect to indices permutation which makes unapplicable the conventional Voigt's notation of contracted indices (Thurston, 1965). However; in practical situations of seismic exploration (for which the magnitude of the components of the stress deviator is much smaller than the magnitude of the wave moduli), all these subtelties can be ignored, the perfect equivalence is practically re-established, and Voigt notation can be used.
– In this latter context, pre-stressed (initially isotropic) solids belong to a special class of orthorhombic (ORT) media, satisfying 3 conditions of ellipticity in the eigen planes of stress (Eq. (19)) and thus characterized by only six instead of nine independent elastic constants (Nikitin and Chesnokov, 1981 and 1984).
– Given the state of pre-stress, the strength of the *P*- or *S*-wave anisotropy and of the *S*-wave birefringence (but not the magnitude of the wave moduli themselves) are determined by only two intrinsic parameters of the medium, one for the *P*-wave (coefficient A_p defined by Eq. (21)) and one for the *S*-waves (coefficient B_s defined by Eq. (23)).
– Nearly exhaustive experimental data on seismic anisotropy in rocks (considered as transversely isotropic) by Thomsen (1986) show that elliptical anisotropy is more an exception than a rule. Since stress-induced anisotropy is essentially elliptical when restricted to the class of transversely isotropic media, as a consequence this work clearly shows that stress can be practically excluded as a unique direct cause of elastic anisotropy in rocks.

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